# Introduction to dynamical systems with applications to biology

Lecture 1

September 26, 2018.

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- Bifurcations and bifurcation diagrams.

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- Mathematics can help in unraveling the complexity in biological systems.

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- Analytical solutions can only be obtained for simple examples.
- For more complex models, simulations are necessary to understand the dynamical behavior.

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- The problem of finding the solution of an ODE with a certain initial condition is called an Initial Value Problem (IVP).

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  - How does the limiting dynamics depend on the initial condition $x_0$?

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  - Does the dynamics display bifurcation? One type of limiting behavior for $\theta < \theta_c$ and another type of behavior for $\theta > \theta_c$.

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- It can be shown that this form is fairly general and encompasses all the examples above.

# General Code for Solving IVPs

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% The code of the function "f" goes here...
end
```

- Main script file:

```
% The code of the main file goes here...
...
% Calling the differential equation solver "ode45"
[t, x] = ode45(@<function_name>, [<Time_Span>], [<Initial_Conditions>]);
% Plotting the solution
plot(t,x);
...
```

# Example 1: Linear IVP

$$\dot{x} = -\theta x \; ; \qquad x(0) = x_0, \qquad \left( f(x) = \theta x, \quad , x \in \mathbb{R} \right)$$

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- Function file (e.g. $\theta = -1$):

```
function [fx] = Linear_Function(t,x)
theta = -1;
fx = theta*x;
end
```

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```
function [fx] = Linear_Function(t,x)
theta = -1;
fx = theta*x;
end
```

- Main script file (e.g. $x_0 = 5$):

```
%% Setting the Initial Condition and Time Span
x0 = 5; Time_Span = [0, 10];

%% Solving...
[t, x] = ode45(@Linear_Function, Time_Span, x0);

%% Plotting the Solution
plot(t, x);
```

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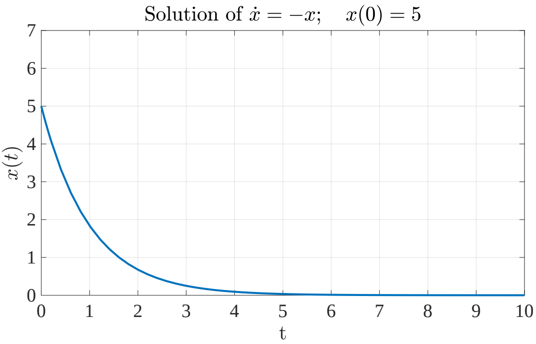

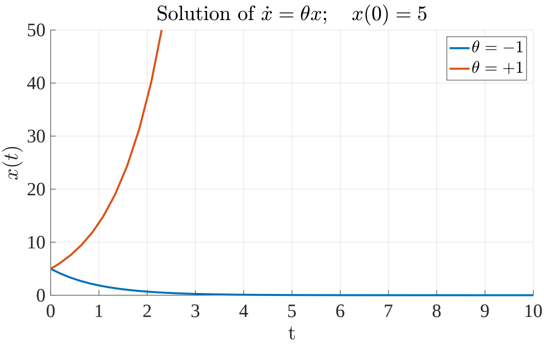


- $\theta < 0 \implies$ stable $\qquad \theta > 0 \implies$ unstable.
- Around $\theta = 0$, a VERY small change can cause the solution to be dramatically different!

# Example 2: Nonlinear IVP

$$\dot{x} = x - x^3\,; \qquad x(0) = x_0, \qquad \left( f(x) = x - x^3, \quad , x \in \mathbb{R} \right)$$

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```
function [fx] = Nonlinear_Function(t,x)
fx = x - x^3;
end
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- Function file:

```
function [fx] = Nonlinear_Function(t,x)
fx = x - x^3;
end
```

- Main script file (e.g. $x_0 = 0.1$):

```
%% Setting the Initial Condition and Time Span
x0 = 0.1; Time_Span = [0, 10];

%% Solving...
[t, x] = ode45(@Nonlinear_Function, Time_Span, x0);

%% Plotting the Solution
plot(t, x);
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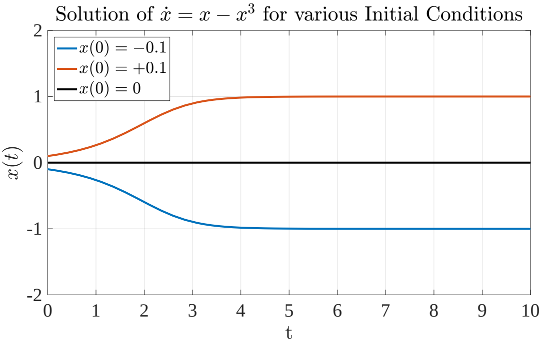

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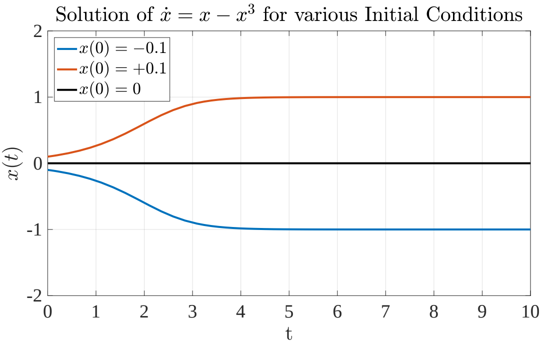


**Observe** : $\lim_{t\to\infty} x(t) = -1$, or $+1$ or 0.

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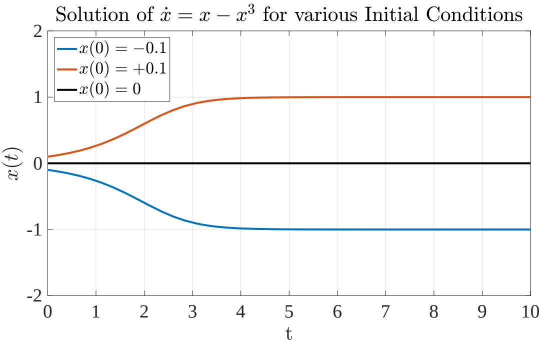


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These are called the fixed points of the dynamical system $\dot{x} = x - x^3$.

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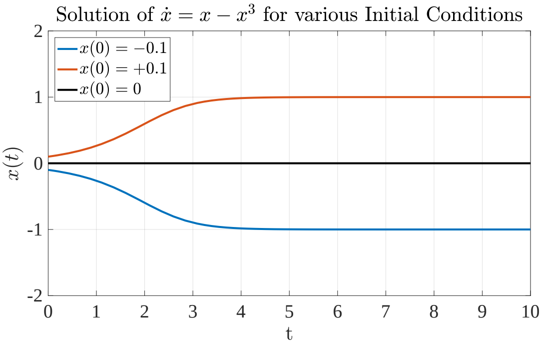


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Fixed points can be calculated analytically *without a simulation*!

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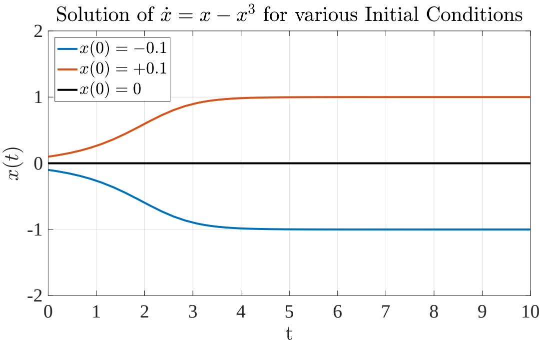


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**Fact**: Fixed points of $\dot{x} = f(x)$ are simply the roots of $f(x) = 0$.

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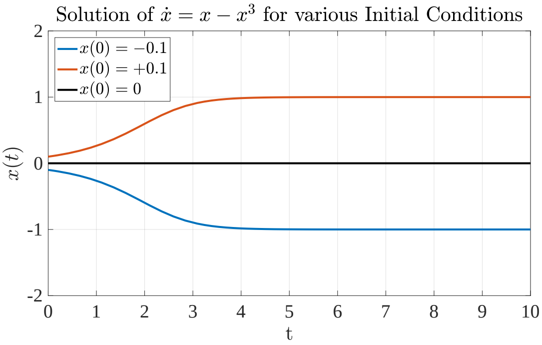


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Fixed points can be calculated analytically *without a simulation*!

**Fact**: Fixed points of $\dot{x} = f(x)$ are simply the roots of $f(x) = 0$.

**In our example**: the roots of $x - x^3 = 0$ are {-1, +1, 0}.

# Example 3: Higher Order IVP

- **Goal**: Solve the following second order IVP using MATLAB

$$\ddot{y} - 5(1 - y^2)\dot{y} + y = 0; \qquad y(0) = 3, \quad \dot{y}(0) = 1.$$

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  3. Rewrite in vector form:
     $$x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad f(x) := \begin{bmatrix} x_2 \\ 5(1-x_1^2)x_2 - x_1 \end{bmatrix} \qquad x_0 := \begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix}$$
     $$\Longrightarrow \boxed{\dot{x} = f(x); \qquad x(0) = x_0}$$

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- Function file:

```
function [fx] = HO_Function(t,x)
fx = [x(2); 5 * (1-x(1)^2) * x(2) - x(1)];
end
```

- Main script file:

```
%% Setting the Initial Condition and Time Span
x0 = [3;1]; Time_Span = [0, 100];

%% Solving...
[t, x] = ode45(@HO_Function, Time_Span, x0);

%% Plotting the Solution
plot(t, x(:,1)); hold on; plot(t, x(:,2));
```

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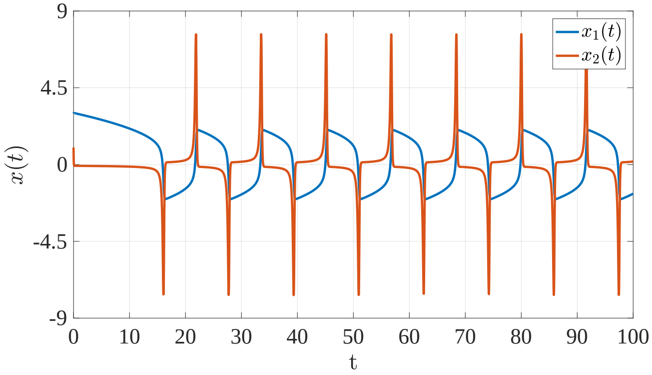

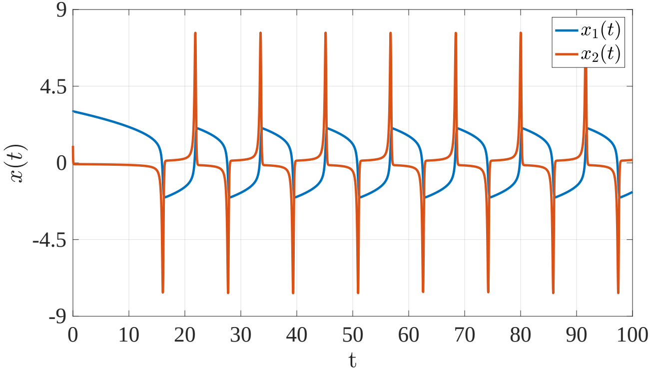


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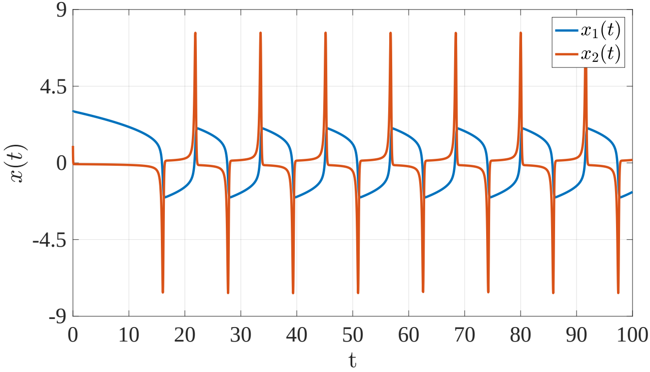


- Solution doesn't converge to a fixed point.
- Solution reaches a limit cycle (periodic oscillations).
- Period around $11s$.

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- Dynamical systems can exhibit a wide range of phenomena (stable, unstable, fixed points, periodic oscillations ...)
- Dynamical systems might be very sensitive to model parameters and/or initial conditions.
- MATLAB can be used to find (numerically) the evolution of dynamical systems.
- In this course, you will learn how to infer some qualitative and quantitative features of dynamical systems by inspection (i.e. without simulations).